Multiple Choice

1 D) $b^2 = a^2(e^2-1)$, since $e = \frac{\sqrt{13}}{3} > 1$

$\therefore e^2 = \frac{b^2}{a^2} + 1 = \frac{13}{9}, \therefore \frac{b^2}{a^2} = \frac{4}{9}$

$\therefore a^2 = 3^2, b^2 = 2^2$

2 A) $\sqrt{7-24i} = \sqrt{16-9-2\times 4\times 3i} = 4-3i$

3 A) double root at 1, quintuple root at -3

4 C) $P'(x) = 3x^2+2x-5, \therefore P(1) = P'(1) = 0$

5 B) $z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

6 C) $u = x^2, du = 2xdx; dv = \sin x dx, v = -\cos x$

$I = uv - \int v du = -x^2\cos x + \int 2x\cos x dx$

7 A) either left or right

8 B) $y^2 = f(x) \Leftrightarrow y = \pm\sqrt{f(x)}$

9 D) From i to the centre 1 is $\sqrt{2}$ plus the radius 1

10 C) $= 1+2+...+101 = \frac{101}{2}(1+101) = 5151$

Question 11

(a) $\frac{4+3i}{2-i} = \frac{(4+3i)(2+i)}{(2-i)(2+i)} = \frac{5+10i}{5} = 1+2i$

(b)(i) $|z| = |-\sqrt{3}+i| = 2$

(ii) $\arg(z) = \frac{5\pi}{6}$

(iii) $\arg\frac{z}{w} = \arg z - \arg w = \frac{5\pi}{6} - \frac{\pi}{7} = \frac{29\pi}{42}$

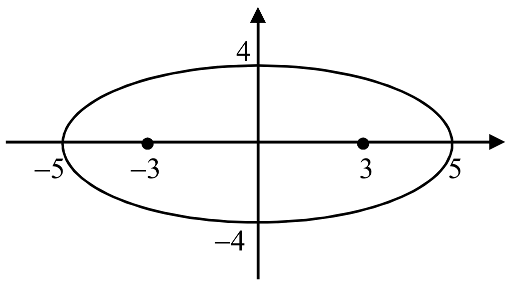
(c) $\frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$

$A = \lim_{x\to 0}\frac{1}{x^2+2} = \frac{1}{2}$

$B = -\frac{1}{2}$, by equating the coefficients of x^2

$C = 0$, by equating the constants

(d) Foci $\left(\pm\sqrt{a^2-b^2}, 0\right) = (\pm 3, 0)$



(e) $x + x^2y^3 = -2$

$1 + 2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-1-2xy^3}{3x^2y^2}$

At $(2,-1), \frac{dy}{dx} = \frac{-1+4}{12} = \frac{1}{4}$

(f) (i) $\cot\theta + \text{cosec}\,\theta = \frac{\cos\theta+1}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \cot\frac{\theta}{2}$

(ii) $\int(\cot\theta + \text{cosec}\,\theta)d\theta = \int\cot\frac{\theta}{2}d\theta$

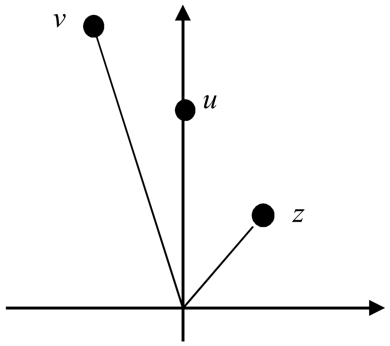
$= \int\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}d\theta = 2\ln\left(\sin\frac{\theta}{2}\right) + C$

Question 12

(a) (i) $z = 2\,\text{cis}\dfrac{\pi}{4} = \sqrt{2}(1+i)$

(ii) $u = z^2 = 4\,\text{cis}\dfrac{\pi}{2} = 4i$

(iii) $v = z^2 - \overline{z} = 4i - \sqrt{2}(1-i) = -\sqrt{2} + (4+\sqrt{2})i$



(b) (i) Since all coefficients are real, the roots are $a \pm ib$ and $a \pm 2ib$.

$\sum \alpha = 4a = 4, \therefore a = 1$

$\prod \alpha = (a^2+b^2)(a^2+4b^2) = (1+b^2)(1+4b^2)$

$= 1 + 5b^2 + 4b^4 = 10, \therefore 4b^4 + 5b^2 - 9 = 0$

$(4b^2+9)(b^2-1) = 0, \therefore b = 1$, since b is real and positive.

$\therefore$ Roots are $1 \pm i, 1 \pm 2i$

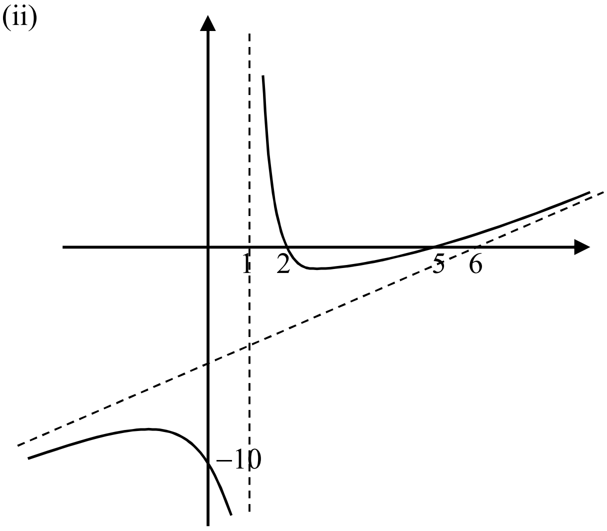
(ii) $P(x) = (x-1-i)(x-1+i)(x-1-2i)(x-1+2i)$

$= \left(x^2-2x+2\right)\left(x^2-2x+5\right)$

(c) (i) $\dfrac{(x-2)(x-5)}{x-1} = \dfrac{x^2-7x+10}{x-1} = x - 6 + \dfrac{4}{x-1}$

Equation of asymptote $y = x - 6$

(ii)



(d) $\partial V = 2\pi R h \partial x$, where $R = (3-x), h = y = \sqrt{x+1}$

$V = 2\pi \int_0^3 (3-x)\sqrt{x+1}\,dx$

Let $u^2 = x+1, 2udu = dx$ and $x = u^2 - 1, \therefore 3 - x = 4 - u^2$

When $x = 0, u = 1$; when $x = 3, u = 2$

$V = 2\pi \int_1^2 (4-u^2)u.2udu$

$= 4\pi \int_1^2 \left(4u^2 - u^4\right)du$

$= 4\pi \left[\dfrac{4u^3}{3} - \dfrac{u^5}{5}\right]_1^2$

$= 4\pi \left(\dfrac{64}{15} - \dfrac{17}{15}\right) = \dfrac{188\pi}{15}$ u^3

Question 13

(a) (i) Sub $Q(a\tan\theta, b\sec\theta)$ to H_2

$\text{LHS} = \tan^2\theta - \sec^2\theta = -1 = \text{RHS}, \therefore Q \in H_2$

(ii) $m = \dfrac{b(\sec\theta - \tan\theta)}{a(\tan\theta - \sec\theta)} = -\dfrac{b}{a}$

$$y - b\sec\theta = -\frac{b}{a}(x - a\tan\theta)$$

$$ay - ab\sec\theta = -bx + ab\tan\theta$$

$$bx + ay = ab(\tan\theta + \sec\theta)$$

(iii) $\text{Area}(\Delta OPQ) = \dfrac{1}{2} PQ \times d$, where $d =$ Perpendicular distance from O to PQ.

$$PQ = |\sec\theta - \tan\theta|\sqrt{b^2 + a^2}$$

$$d = \frac{|ab(\tan\theta + \sec\theta)|}{\sqrt{a^2 + b^2}}$$

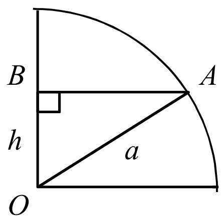
$$\therefore \text{Area}(\Delta OPQ) = \frac{ab\sqrt{b^2 + a^2}\left(\sec^2\theta - \tan^2\theta\right)}{\sqrt{a^2 + b^2}}$$

$= ab$, which is independent of θ.

(b) (i) $AB^2 = OA^2 - OB^2 = a^2 - h^2$

$\therefore AB = \sqrt{a^2 - h^2}$

(can't think of a better way to do this question)



(ii) $\partial V = AB^2 \times \partial h$

$$V = \int_0^a (a^2 - h^2)dh = \left[a^2 h - \frac{h^3}{3}\right]_0^a = \frac{2a^3}{3} \text{ u}^3$$

(c) (i) $S = 4\pi r^2, \therefore \dfrac{dS}{dr} = 8\pi r$

$$\frac{dr}{dt} = \frac{dr}{dS}\frac{dS}{dt} = \frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$$

(ii) $V = \dfrac{4}{3}\pi r^3, \therefore \dfrac{dV}{dr} = 4\pi r^2$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \frac{4\pi r^2}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} = \frac{r}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$$

$$= \frac{1}{2}\left(\frac{4\pi r^3}{3}\right)^{\frac{1}{3}} = \frac{1}{2}V^{\frac{1}{3}}$$

(iii) $\dfrac{dV}{dt} = \dfrac{1}{2}V^{\frac{1}{3}}, \therefore \displaystyle\int_{8000}^{64000} 2V^{-\frac{1}{3}}dV = \int_0^T dt$

$$\therefore T = 2\left[\frac{3V^{\frac{2}{3}}}{2}\right]_{8000}^{64000} = \left[3V^{\frac{2}{3}}\right]_{8000}^{64000} = 3600 \text{ sec.}$$

Question 14

(a) (i) $\dfrac{d}{d\theta}\sin^{n-1}\theta\cos\theta = (n-1)\sin^{n-2}\theta\cos^2\theta - \sin^n\theta$

$$= (n-1)\sin^{n-2}\theta\left(1 - \sin^2\theta\right) - \sin^n\theta$$

$$= (n-1)\sin^{n-2}\theta - (n-1)\sin^n\theta - \sin^n\theta$$

$$= (n-1)\sin^{n-2}\theta - n\sin^n\theta$$

(ii) $\left[\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} = (n-1)\displaystyle\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta$

$-n\displaystyle\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta$, by integrating both sides of the result of part (i) with respect to θ, from 0 to $\dfrac{\pi}{2}$,

$$0 = (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta - n\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta.$$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^n\theta d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta.$$

(iii) Let $I_4 = \displaystyle\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta$

$$I_4 = \frac{3}{4}I_2$$

$$I_2 = \frac{1}{2}I_0$$

$$I_0 = \int_0^{\frac{\pi}{2}}d\theta = \frac{\pi}{2}.$$

$$\therefore I_4 = \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3\pi}{16}.$$

(b) (i) $p = \sum\alpha\beta$.

$$(\alpha + \beta + \gamma)^2 = \sum\alpha^2 + 2\sum\alpha\beta$$

$$0 = 16 + 2(-p)$$

$$\therefore p = 8.$$

(ii) $x^3 = px - q$

$$\sum\alpha^3 = p\sum\alpha - 3q$$

$$-9 = 0 - 3q$$

$$\therefore q = 3$$

(iii) $x^4 = px^2 - qx$

$$\sum\alpha^4 = p\sum\alpha^2 - q\sum\alpha = 16p - 0 = 16\times 8 = 128$$

(c) (i) Resolving the forces

vertically, $N\cos\theta - \mu N\sin\theta = mg$ (1)

horizontally, $N\sin\theta + \mu N\cos\theta = \dfrac{mv^2}{r}$ (2)

$\dfrac{(2)}{(1)}$ gives $\dfrac{v^2}{rg} = \dfrac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}$.

$v^2 = rg\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}$

(ii) When $V^2 = rg$, if the tendency to slide up the track still exists, then $\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta} = 1$

$\tan\theta + \mu = 1 - \mu\tan\theta$

$\mu(1 + \tan\theta) = 1 - \tan\theta$

$\mu = \dfrac{1 - \tan\theta}{1 + \tan\theta} = 1 - \dfrac{2\tan\theta}{1 + \tan\theta}$

For $0 < \theta < \dfrac{\pi}{2}, \tan\theta > 0, \therefore \dfrac{2\tan\theta}{1 + \tan\theta} > 0, \therefore \mu < 1$

Question 15

(a) (i) $a = \dfrac{dv}{dt} = -kv^2$

$\displaystyle\int \frac{dv}{v^2} = -k\int dt$

$-\dfrac{1}{v} = -kt + C$

When $t = 0, v = u, \therefore C = -\dfrac{1}{u}$

$\therefore \dfrac{1}{v} = kt + \dfrac{1}{u}$.

(ii) $a = \dfrac{dw}{dt} = -kw^2 - g$

$\displaystyle\int \frac{dv}{g + kw^2} = -\int dt$

$\dfrac{1}{\sqrt{gk}}\tan^{-1} w\sqrt{\dfrac{k}{g}} = -t + C$

When $t = 0, w = u, \therefore C = \dfrac{1}{\sqrt{gk}}\tan^{-1} u\sqrt{\dfrac{k}{g}}$

$t = \dfrac{1}{\sqrt{gk}}\left(\tan^{-1} u\sqrt{\dfrac{k}{g}} - \tan^{-1} w\sqrt{\dfrac{k}{g}}\right)$

(iii) When $w = 0, t = \dfrac{1}{\sqrt{gk}}\tan^{-1} u\sqrt{\dfrac{k}{g}}$.

Sub to the result of (i), letting $v = V$

$\dfrac{1}{V} = \dfrac{k}{\sqrt{gk}}\tan^{-1} u\sqrt{\dfrac{k}{g}} + \dfrac{1}{u}$

$= \dfrac{1}{u} + \sqrt{\dfrac{k}{g}}\tan^{-1} u\sqrt{\dfrac{k}{g}}$

(iv) When $u \to \infty, \dfrac{1}{u} \to 0, \tan^{-1} u\sqrt{\dfrac{k}{g}} \to \dfrac{\pi}{2}$

$\therefore \dfrac{1}{V} \to \dfrac{\pi}{2}\sqrt{\dfrac{k}{g}}$

$\therefore V \to \dfrac{2}{\pi}\sqrt{\dfrac{g}{k}}$

(b) (i) If $x \geq 0,\ 1-x^2 \leq 1, \therefore (1-x)(1+x) \leq 1$

$\therefore 1-x \leq \frac{1}{1+x}$, since $1+x>0$

Also, $\frac{1}{1+x} \leq 1$ since the denominator > numerator

$\therefore 1-x \leq \frac{1}{1+x} \leq 1$

(ii) $\int_0^{\frac{1}{n}} (1-x)dx \leq \int_0^{\frac{1}{n}} \frac{1}{1+x}dx \leq \int_0^{\frac{1}{n}} dx$

$\left[x-\frac{x^2}{2}\right]_0^{\frac{1}{n}} \leq \left[\ln(1+x)\right]_0^{\frac{1}{n}} \leq \frac{1}{n}$

$\frac{1}{n}-\frac{1}{2n^2} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$

$1-\frac{1}{2n} \leq n\ln\left(1+\frac{1}{n}\right) \leq 1$

(iii) When $n \to \infty, \frac{1}{2n} \to 0, \therefore n\ln\left(1+\frac{1}{n}\right) \to 1$

(This is the sandwich theorem: if $g(x) \leq f(x) \leq h(x)$ and $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} f(x) = L$)

$\therefore \ln\left(1+\frac{1}{n}\right)^n \to 1 = \ln e$

$\therefore \left(1+\frac{1}{n}\right)^n \to e$

(c) (i) If $x,y>0, x+y=\sqrt{(x+y)^2} = \sqrt{x^2+y^2+2xy}$

$\therefore x+y \leq \sqrt{x^2+y^2}$

$\therefore \sqrt{xy} \leq \frac{x+y}{2} \leq \frac{\sqrt{x^2+y^2}}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$

(ii) $\sqrt{ab} \leq \sqrt{\frac{a^2+b^2}{2}}$

$\sqrt{cd} \leq \sqrt{\frac{c^2+d^2}{2}}$

$\sqrt{\sqrt{ab}\sqrt{cd}} \leq \sqrt{\frac{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}}{2}}$, as $\sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$

$\sqrt[4]{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

Question 16

(a) (i) The number of ways to place a black counter on each column $= 3, \therefore$ for 5 columns, it's 3^5.

Total number of ways $= {}^{15}C_5 \times {}^{10}C_{10} = {}^{15}C_5$, since there are ${}^{15}C_5$ ways to place 5 black counters in the 15 cells and ${}^{10}C_{10}$ ways to place the 10 white cells.

$\therefore \Pr = \frac{3^5}{{}^{15}C_5} = \frac{243}{3003} = \frac{81}{1001}$

(ii) If there are n rows and q columns, then there are n ways to put a black counter in each column, $\therefore$ for q columns, it's n^q.

Similar to part (i), the total number of ways to place q black cells in nq cells is ${}^{nq}C_q \times {}^{nq-q}C_{nq-q} = {}^{nq}C_q$.

$\therefore \Pr = \frac{n^q}{{}^{nq}C_q}$.

(iii) $\Pr = \frac{n^q q!(nq-q)!}{(nq)!} = \frac{n^q q!}{(nq)(nq-1)...(nq-q+1)}$

$= \frac{n^q q!}{n^q\left(q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)...\left(q-\frac{q-1}{n}\right)\right)}$

$= \frac{q!}{q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)...\left(q-\frac{q-1}{n}\right)}$.

Thus, when $n \to \infty, \Pr \to \frac{q!}{q^q}$.

(b) (i) Let $\cos\alpha = c, \sin\alpha = s$

$(c+is)^{2n} = c^{2n} + \binom{2n}{1}c^{2n-1}(is) + \binom{2n}{2}c^{2n-2}(is)^2$

$+\binom{2n}{3}c^{2n-3}(is)^3 + \binom{2n}{4}c^{2n-4}(is)^4 + ...$

$+\binom{2n}{2n-2}c^2(is)^{2n-2} + \binom{2n}{2n-1}c(is)^{2n-1} + (is)^{2n}$

$= \cos 2n\alpha + i\sin 2n\alpha$, by De Moivre's theorem.

Given $i^2 = -1$, by equating the real parts

$\cos 2n\alpha = \cos^{2n}\alpha - \binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha$

$+\binom{2n}{4}\cos^{2n-4}\alpha\sin^4\alpha + ... + (-1)^n\sin^{2n}\alpha.$

(ii) Let $\alpha = \cos^{-1} x, \cos(\cos^{-1} x) = x, \sin^2 \alpha = 1 - \cos^2 \alpha = 1 - x^2$,

$\cos\left(2n\cos^{-1} x\right) = x^{2n}\alpha - \binom{2n}{2} x^{2n-2}(1-x^2)$

$+\binom{2n}{4} x^{2n-4}(1-x^2)^2 + \ldots + (-1)^n (1-x^2)^n$

(iii) Let $\cos(2n\cos^{-1} x) = 0$

$2n\cos^{-1} x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$

$\cos^{-1} x = \frac{(2k+1)\pi}{4n}$

$x = \cos\frac{(2k+1)\pi}{4n}, k = 0,1,2,\ldots,2n-1$

$x = \cos\frac{\pi}{4n}, \cos\frac{3\pi}{4n}, \ldots, \cos\frac{(4n-1)\pi}{4n}$.

$\prod \alpha = \cos\frac{\pi}{4n}\cos\frac{3\pi}{4n}\ldots\cos\frac{(4n-1)\pi}{4n}$

$= \frac{(-1)^{2n}(-1)^n}{\text{coefficient of } x^{2n}}$

$= \frac{(-1)^n}{1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}}$. (1)

$(1+x)^{2n} = 1 + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{2n}x^{2n}$ (2)

Let $x = 1$, $2^{2n} = 1 + \binom{2n}{1} + \binom{2n}{2} + \ldots + \binom{2n}{2n}$

Let $x = -1$, $0 = 1 - \binom{2n}{1} + \binom{2n}{2} - \ldots + \binom{2n}{2n}$

$\therefore 2^{2n} = 2\left(1 + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n}\right)$

$\therefore 1 + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}$.

$\therefore$ From (1), $\cos\frac{\pi}{4n}\cos\frac{3\pi}{4n}\ldots\cos\frac{(4n-1)\pi}{4n} = \frac{(-1)^n}{2^{2n-1}}$

(iv) From $\cos(2n\cos^{-1} x) = x^{2n} - \binom{2n}{2}x^{2n-2}(1-x^2)$

$+\binom{2n}{4}x^{2n-4}(1-x^2)^2 + \ldots + (-1)^n(1-x^2)^n$, let $x = \frac{1}{\sqrt{2}}$,

$\text{LHS} = \cos\left(2n \times \frac{\pi}{4}\right) = \cos\frac{n\pi}{2}$

$\text{RHS} = \frac{1}{2^n}\left(1 - \binom{2n}{2} + \binom{2n}{4} - \ldots + (-1)^n\binom{2n}{2n}\right)$

$\therefore 1 - \binom{2n}{2} + \binom{2n}{4} - \ldots + (-1)^n\binom{2n}{2n} = 2^n\cos\frac{n\pi}{2}$.

Alternatively, from (2), let $x = i$ and $x = -i$ respectively,

$(1+i)^{2n} = 1 + \binom{2n}{1}i - \binom{2n}{2} - \binom{2n}{3}i + \ldots + (-1)^n\binom{2n}{2n}$

$(1-i)^{2n} = 1 - \binom{2n}{1}i - \binom{2n}{2} + \binom{2n}{3}i + \ldots + (-1)^n\binom{2n}{2n}$

$\therefore (1+i)^{2n} + (1-i)^{2n} = 2\left(1 - \binom{2n}{2} + \binom{2n}{4} - \ldots\right.$

$\left. + (-1)^n\binom{2n}{2n}\right)$

$\therefore 1 - \binom{2n}{2} + \binom{2n}{4} - \ldots + (-1)^n\binom{2n}{2n}$

$= \frac{(1+i)^{2n} + (1-i)^{2n}}{2}$

$= \frac{1}{2}\left(\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^{2n} + \left(\sqrt{2}\,\text{cis}\frac{-\pi}{4}\right)^{2n}\right)$

$= \frac{2^n}{2}\left(\text{cis}\frac{n\pi}{2} + \text{cis}\frac{-n\pi}{2}\right)$

$= \frac{2^n}{2} \times 2\cos\frac{n\pi}{2}$

$= 2^n\cos\frac{n\pi}{2}$.